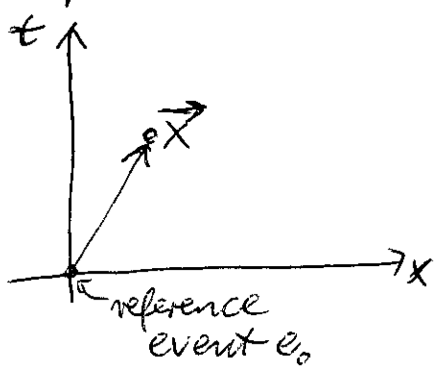
Tensor analysis in SR, part I

<u>Def.</u>: a vector is an element of a $\underbrace{\text{vector space}}$
↳ defined as usual

also as usual: decomposition in basis vectors $\vec{e}_\mu$, $\mu = 0,1,2,3$ (↑ time)

example: coordinate vector



$$\vec{x} = \sum_{\mu=0}^{3} x^\mu \vec{e}_\mu = x^\mu \vec{e}_\mu$$

↑ Einstein's summation convention

↑ components in basis $\vec{e}_\mu$: $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

<u>but</u>: $\vec{x}$ is independent of the choice of basis! [no basis in def. of vector]

<u>Def</u>: <u>linear form</u> or 1-form f is a linear map from vector to number

$$f: \text{vector} \longrightarrow \text{number}, \quad f(a\vec{x} + b\vec{y}) = a f(\vec{x}) + b f(\vec{y})$$

space of linear forms: (dual) vector space
↳ choose a basis: require $\omega^\mu(\vec{e}_\nu) = \delta^\mu_\nu$ ⇝ ω^μ dual basis of $\vec{e}_\mu$
↳ basis decomposition: $f = f_\mu \omega^\mu$ (↑ components, ↑ basis)

then: $\underline{f(\vec{x})} = f_\mu \omega^\mu(\vec{x}) = f_\mu \omega^\mu(x^\nu \vec{e}_\nu)$

$$= f_\mu x^\nu \underbrace{\omega^\mu(\vec{e}_\nu)}_{\delta^\mu_\nu} = \underline{\underline{f_\mu x^\mu}}$$

makes sense [form maps vector and vice versa, generalization?]

<u>Def</u>: a (p,q)-<u>tensor</u> is a <u>linear map</u> from p linear forms and q vectors to a number: $T(f_1, \ldots, f_p, \vec{x}_1, \ldots, \vec{x}_q) = \text{number}$

↳ geometric object, no basis or components needed to define it!
↳ basis decomposition:

$$T(f_1, \ldots, f_p, \vec{x}_1, \ldots \vec{x}_q) = T(f_{1\mu_1}\omega^{\mu_1}, \ldots, f_{p\mu_p}\omega^{\mu_p}, x_1^{\nu_1}\vec{e}_{\nu_1}, \ldots, x_q^{\nu_q}\vec{e}_{\nu_q})$$

$$= f_{1\mu_1} \cdots f_{p\mu_p} x_1^{\nu_1} \cdots x_q^{\nu_q} \underbrace{T(\omega^{\mu_1}, \ldots, \omega^{\mu_p}, \vec{e}_{\nu_1}, \ldots, \vec{e}_{\nu_q})}_{T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}}$$

$T^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}$: components of T in basis $\vec{e}_\nu$, ω^μ

more definitions:

- upper indices: ~~K~~ contravariant indices
  lower " : covariant "
- tensor field: map of spacetime point to tensor
- n-form: completely antisymmetric $(0,n)$-tensor
- differential form: n-form field

<u>example</u>: scalar product $\vec{X}\cdot\vec{Y} \equiv g(\vec{X},\vec{Y})$ = number, is linear

$\rightsquigarrow g$ is a $(0,2)$-tensor $\rightsquigarrow$ metric tensor

symmetric $g(\vec{X},\vec{Y}) = g(\vec{Y},\vec{X}) \rightsquigarrow g_{\mu\nu} = g_{\nu\mu}$

inverse metric $g^{\mu\nu}$: inverse matrix of $g_{\mu\nu}$

$\hookrightarrow g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$

"pulling indices": $X_\mu \equiv g_{\mu\nu} X^\nu$, $f^\nu \equiv g^{\nu\mu} f_\mu$
convention

bijection vectors $\longleftrightarrow$ 1-forms
analogous for tensors

<u>orthonormal Lorentz basis</u>:

one can choose a basis such that

$$g(\vec{e}_\mu, \vec{e}_\nu) = g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \equiv \eta_{\mu\nu}$$ Minkowsky metric

(diagonalization of ~~g~~ symmetric matrix & rescaling)

$$\rightsquigarrow \vec{X}\cdot\vec{Y} = \eta_{\mu\nu} X^\mu Y^\nu = -X^0Y^0 + \underbrace{X^1Y^1 + X^2Y^2 + X^3Y^3}_{\text{Euclidean scalar product}}$$

writing $(X^\mu) = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$: $\vec{X}\cdot\vec{X} = -t^2 + x^2 + y^2 + z^2$ = invariant $\forall$ (lecture 1) under change of inertial system with same origin

<u>line element</u>: $\Delta\tau^2 = -g_{\mu\nu}\Delta X^\mu \Delta X^\nu$

or $\boxed{d\tau^2 = -g_{\mu\nu} dX^\mu dX^\nu}$

Lorentz trafo:

change of inertial system must leave scalar product/metric invariant ⇝ isometry

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (*) \quad \text{such that} \quad \eta_{\mu\nu} x'^\mu x'^\nu = \eta_{\mu\nu} x^\mu x^\nu \quad \forall \vec{x}$$

↑ Lorentz matrix

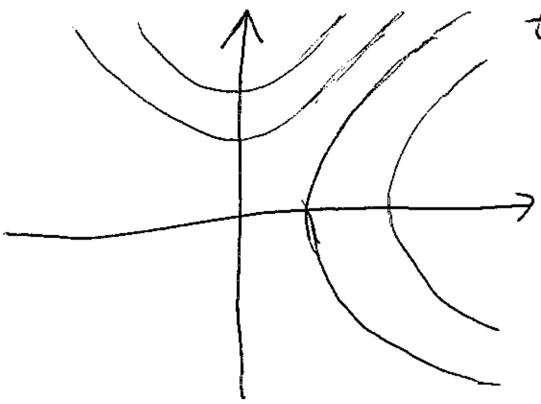
$$\rightsquigarrow \boxed{\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}}$$

compare to rotations

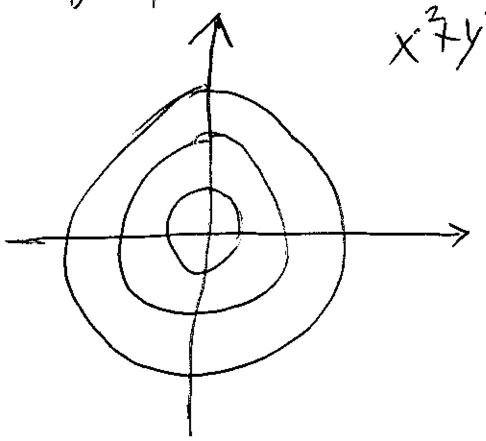
$$\delta_{ij} R^i{}_k R^j{}_l = \delta_{kl} \quad \text{OR} \quad R^T R = \mathbb{1}$$

Lorentz trafos are "rotations" along hyperbola

$t^2 - x^2 = \text{const}$



$x^2 + y^2 = \text{const}$



$t' = \cosh\alpha \cdot t - \sinh\alpha \cdot x$

$x' = -\sinh\alpha \cdot t + \cosh\alpha \cdot x$

$\rightsquigarrow t'^2 - x'^2 = t^2 - x^2$

$v = \tanh\alpha = \frac{x'}{t'}$ for $x = 0$

$\rightsquigarrow \cosh\alpha = \gamma, \ \sinh\alpha = v \cdot \gamma$

$\rightsquigarrow t' = \gamma t - \gamma v x \qquad y' = y$

$x' = -\gamma v t + \gamma x \qquad z' = z$ } compare to (*)

$$\left(\Lambda^\mu{}_\nu\right) = {}^{\mu\downarrow}\overset{\nu\rightarrow}{\begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & 0 \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}}$$

Lorentz matrix for boost in x-direction

$x' = \cos\theta \cdot x + \sin\theta \cdot y$

$y' = -\sin\theta \cdot y + \cos\theta \cdot x$

$\rightsquigarrow x'^2 + y'^2 = \cancel{\text{const}}\ x^2 + y^2$

Why tensors?

inertial frames $\hat{=}$ orthonormal basis

principle of special relativity

↳ laws of electrodyn./laws of physics in inertial frames should be written as tensor equations on spacetime

⇝ geometry of tensors on spacetime is manifestly independent of a specific basis/inertial frame!