EFT calculation of the gravitational binary effective action at 1PN

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1 Literature

Effective Field Theory (EFT) program to classical gravitational dynamics and radiation was initiated in references [1, 2], for a review of the literature see references [3–5]. The first post-Newtonian (1PN) Lagrangian was first derived in reference [1] using the EFT approach, here we follow reference [6]. A Mathematica code was published in reference [7].

2 Feynman rules

2.1 Propagators

$$
\begin{aligned}
\text{———} &= \langle \phi(\boldsymbol{x}_1, t_1)\phi(\boldsymbol{x}_2, t_2)\rangle, && (1)\\
&= 4\pi\delta(t_1 - t_2)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{k}^2}, && (2)\\
\text{- - - - -} &= \langle A_i(\boldsymbol{x}_1, t_1)A_j(\boldsymbol{x}_2, t_2)\rangle, && (3)\\
&= -\pi c^2\delta_{ij}\delta(t_1 - t_2)\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{\boldsymbol{k}^2}. && (4)
\end{aligned}
$$

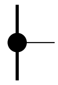
2.2 Field Vertices

$$
\text{———}\bullet\text{———} = \frac{1}{4\pi c^2}\int d^4x\,(\partial_0\phi)^2, \qquad (5)
$$

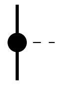
This was visualized by a cross in the lectures, I have not yet figured out how to do this with latex.

2.3 Worldline vertices

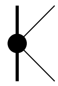
The worldline couplings are to 1PN order



$$= -m_1 \int dt_1\, \phi(\boldsymbol{x}_1(t_1), t_1) \left[1 + \frac{3}{2c^2}\boldsymbol{v}_1^2(t_1)\right], \tag{6}$$



$$= \frac{4m_1}{c^2} \int dt_1\, A_i(\boldsymbol{x}_1(t_1), t_1)\, v_1^i(t_1), \tag{7}$$



$$= -\frac{m_1}{c^2} \int dt_1\, [\phi(\boldsymbol{x}_1(t_1), t_1)]^2, \tag{8}$$

where $\boldsymbol{v}_1 = \dot{\boldsymbol{x}}_1 = d\boldsymbol{x}_1/dt$. The worldlines are represented by thick lines, but note that there are no propagators associated to them. The field-independent parts of the action

$$m_1 \int dt \left[-c^2 + \frac{1}{2}\boldsymbol{v}_1^2 + \frac{1}{8c^2}\boldsymbol{v}_1^4\right], \tag{9}$$

can just be taken over to S_{eff} unchanged, one copy for each body. **Note that there was a typo in the lectures in the last term here.**

3 Integrals

A useful integral is

$$\int \frac{d^d\boldsymbol{k}}{(2\pi)^d} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{(\boldsymbol{k}^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2-\alpha)}{\Gamma(\alpha)} \left(\frac{\boldsymbol{x}^2}{4}\right)^{\alpha-d/2}. \tag{10}$$

Show this using Schwinger parametrization. The special cases $d = 3$, $\alpha = 1, 2$ read

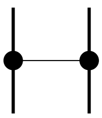
$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^2} = \frac{1}{4\pi|\boldsymbol{x}|} \tag{11}$$

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{x}}}{\boldsymbol{k}^4} = -\frac{|\boldsymbol{x}|}{8\pi}. \tag{12}$$

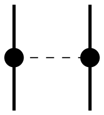
4 1PN diagrams

The Feynman diagrams can be translated into integrals by "pasting" in the right-hand sides of the Feynman rules; fields are contracted using $\langle \ldots \rangle$ according to the propagator lines.

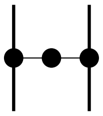
Compute the diagrams/integrals below!



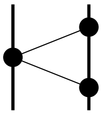
$$= \int dt_1\, m_1 \left[1 + \frac{3}{2c^2}\boldsymbol{v}_1^2(t_1)\right] \int dt_2\, m_2 \left[1 + \frac{3}{2c^2}\boldsymbol{v}_2^2(t_2)\right] \langle \phi(\boldsymbol{x}_1(t_1), t_1)\phi(\boldsymbol{x}_2(t_2), t_2)\rangle. \tag{13}$$



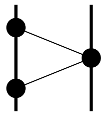
$$= \int dt_1 \frac{4m_1}{c^2} v_1^i(t_1) \int dt_2 \, \frac{4m_2}{c^2} v_2^j(t_2) \langle A_i(\boldsymbol{x}_1(t_1), t_1) A_j(\boldsymbol{x}_2(t_2), t_2) \rangle. \tag{14}$$



$$= \int dt_1 dt_2 d^3\boldsymbol{x} dt \, \frac{m_1 m_2}{4\pi c^2} \, \partial_t \langle \phi(\boldsymbol{x}_1(t_1), t_1) \phi(\boldsymbol{x}, t) \rangle \, \partial_t \langle \phi(\boldsymbol{x}_2(t_2), t_2) \phi(\boldsymbol{x}, t) \rangle. \tag{15}$$

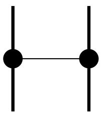


$$= -\frac{1}{2} \int dt_1 dt_2 dt_3 \, \frac{m_1 m_2^2}{c^2} \langle \phi(\boldsymbol{x}_1(t_1), t_1) \phi(\boldsymbol{x}_2(t_2), t_2) \rangle \langle \phi(\boldsymbol{x}_1(t_1), t_1) \phi(\boldsymbol{x}_2(t_3), t_3) \rangle. \tag{16}$$



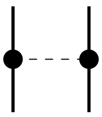
$$= \text{similar}. \tag{17}$$

5 Solution



$$\begin{aligned}
&= \int dt_1 \, m_1 \left[1 + \frac{3}{2} \boldsymbol{v}_1^2(t_1) \right] \int dt_2 \, m_2 \left[1 + \frac{3}{2} \boldsymbol{v}_2^2(t_2) \right] \langle \phi(\boldsymbol{x}_1(t_1), t_1) \phi(\boldsymbol{x}_2(t_2), t_2) \rangle, \\
&\approx m_1 m_2 \, 4\pi \int dt_1 dt_2 \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left[1 + \frac{3}{2} \boldsymbol{v}_1^2(t_1) + \frac{3}{2} \boldsymbol{v}_2^2(t_2) \right] \frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1) - \boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^2} \delta(t_1 - t_2), \\
&= \int dt \, \frac{m_1 m_2}{r(t)} \left[1 + \frac{3}{2} \boldsymbol{v}_1^2(t) + \frac{3}{2} \boldsymbol{v}_2^2(t) \right], \qquad (18)
\end{aligned}$$

where $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$. We have dropped terms higher than quadratic in the velocities, since these would be 2PN. Notice that the result is the Newtonian gravitational potential, with some velocity corrections at 1PN.

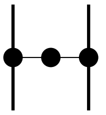


$$= \int dt_1 4m_1 v_1^i(t_1) \int dt_2 \, 4m_1 v_2^i(t_2) \langle A_i(\boldsymbol{x}_1(t_1), t_1) A_j(\boldsymbol{x}_2(t_2), t_2) \rangle, \tag{19}$$

$$= -\int dt_1 dt_2 \frac{d^3\boldsymbol{k}}{(2\pi)^3} \, 16\pi m_1 m_2 v_1^i(t_1) v_2^j(t_2) \delta_{ij} \frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1) - \boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^2}, \tag{20}$$

$$= -\int dt \, \frac{4m_1 m_2}{r} \boldsymbol{v}_1 \cdot \boldsymbol{v}_2, \tag{21}$$

where we have suppressed the time dependence in the final result.



$$= \int dt_1 dt_2 d^3\boldsymbol{x} dt\, \frac{m_1 m_2}{4\pi} \left\langle \phi(\boldsymbol{x}_1(t_1), t_1)\partial_t \phi(\boldsymbol{x}, t)\right\rangle \left\langle \phi(\boldsymbol{x}_2(t_2), t_2)\partial_t \phi(\boldsymbol{x}, t)\right\rangle, \tag{22}$$

$$= 4\pi \int dt_1 dt_2 dt \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3} d^3\boldsymbol{x}\, m_1 m_2 \frac{e^{i\boldsymbol{k}_1\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}]}e^{i\boldsymbol{k}_2\cdot[\boldsymbol{x}_2(t_2)-\boldsymbol{x}]}}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}\, \partial_t\delta(t_1-t)\,\partial_t\delta(t_2-t),$$

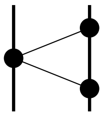
$$= 4\pi \int dt_1 dt_2 dt d^3\boldsymbol{k}_1 \frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\, m_1 m_2 \frac{e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}_1(t_1)}e^{i\boldsymbol{k}_2\cdot\boldsymbol{x}_2(t_2)}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2}\, \partial_{t_1}\delta(t_1-t)\,\partial_{t_2}\delta(t_2-t),$$

$$= 4\pi \int dt_1 dt_2 dt \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, m_1 m_2 \frac{e^{i\boldsymbol{k}\cdot[\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)]}}{\boldsymbol{k}^4}\, \partial_{t_1}\delta(t_1-t)\,\partial_{t_2}\delta(t_2-t), \tag{23}$$

$$= -\int dt_1 dt_2 dt\, \frac{m_1 m_2}{2}\delta(t_1-t)\delta(t_2-t)\partial_{t_1}\partial_{t_2}|\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)|, \tag{24}$$

$$= \int dt\, \frac{m_1 m_2}{2r}\left(\boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \boldsymbol{v}_1\cdot\boldsymbol{n}\,\boldsymbol{v}_2\cdot\boldsymbol{n}\right), \tag{25}$$

where $\boldsymbol{n} = (\boldsymbol{x}_1 - \boldsymbol{x}_2)/r$.

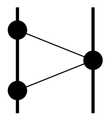


$$= -\frac{1}{2}\int dt_1 dt_2 dt_3\, m_1 m_2^2 \langle\phi(\boldsymbol{x}_1(t_1), t_1)\phi(\boldsymbol{x}_2(t_2), t_2)\rangle\langle\phi(\boldsymbol{x}_1(t_1), t_1)\phi(\boldsymbol{x}_2(t_3), t_3)\rangle, \tag{26}$$

$$= -\frac{1}{2}\int dt_1 dt_2 dt_3\, m_1 m_2^2\, \frac{\delta(t_1-t_2)\delta(t_1-t_3)}{|\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_2)|\,|\boldsymbol{x}_1(t_1)-\boldsymbol{x}_2(t_3)|}, \tag{27}$$

$$= -\int dt\, \frac{m_1^2 m_2}{2r^2}. \tag{28}$$

The last diagram follows from the previous by an exchange of body labels



$$= -\int dt\, \frac{m_1 m_2^2}{2r^2}. \tag{29}$$

6 Result

Summing all results, The action at 1PN finally reads

$$S^{\mathrm{o}}_{\mathrm{eff}} = \int dt \Bigg[-m_1 - m_2 + \frac{m_1}{2}\boldsymbol{v}_1^2 + \frac{m_2}{2}\boldsymbol{v}_2^2 + \frac{1}{8}m_1\boldsymbol{v}_1^4 + \frac{1}{8}m_2\boldsymbol{v}_2^4 + \frac{m_1 m_2}{r}\left(1 + \frac{3}{2}\boldsymbol{v}_1^2 + \frac{3}{2}\boldsymbol{v}_2^2 - \frac{7}{2}\boldsymbol{v}_1\cdot\boldsymbol{v}_2 - \frac{1}{2}\boldsymbol{v}_1\cdot\boldsymbol{n}\,\boldsymbol{v}_2\cdot\boldsymbol{n}\right) - \frac{m_1 m_2(m_1+m_2)}{2r^2}\Bigg], \tag{30}$$

where $\boldsymbol{n} = (\boldsymbol{x}_1 - \boldsymbol{x}_2)/r$.

References

[1] W. D. Goldberger and I. Z. Rothstein, "An Effective field theory of gravity for extended objects," *Phys. Rev.* **D73** (2006) 104029, arXiv:hep-th/0409156 [hep-th].

[2] W. D. Goldberger, "Effective field theories and gravitational radiation," in *Les Houches Summer School - Session 86: Particle Physics and Cosmology: The Fabric of Spacetime*, pp. 351–353, 355–396. 2007. arXiv:hep-ph/0701129 [hep-ph]. http://www.sciencedirect.com/science/bookseries/09248099/86.

[3] S. Foffa and R. Sturani, "Effective field theory methods to model compact binaries," *Class. Quant. Grav.* **31** no. 4, (2014) 043001, arXiv:1309.3474 [gr-qc].

[4] R. A. Porto, "The effective field theorist's approach to gravitational dynamics," *Phys. Rept.* **633** (2016) 1–104, arXiv:1601.04914 [hep-th].

[5] M. Levi, "Effective Field Theories of Post-Newtonian Gravity: A comprehensive review," arXiv:1807.01699 [hep-th].

[6] B. Kol and M. Smolkin, "Non-Relativistic Gravitation: From Newton to Einstein and Back," *Class. Quant. Grav.* **25** (2008) 145011, arXiv:0712.4116 [hep-th].

[7] M. Levi and J. Steinhoff, "EFTofPNG: A package for high precision computation with the Effective Field Theory of Post-Newtonian Gravity," *Class. Quant. Grav.* **34** no. 24, (2017) 244001, arXiv:1705.06309 [gr-qc].