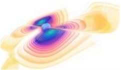
Canonical formulation of spinning objects in General Relativity

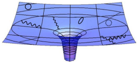
Jan Steinhoff Steven Hergt Gerhard Schäfer

since 1558

Theoretisch-Physikalisches Institut
Friedrich-Schiller-Universität Jena

DPG Spring Meeting, March 19th, 2010, Bonn





DFG: SFB/TR7 “Gravitational Wave Astronomy” and GRK 1523

Outline

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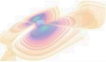

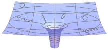

Canonical Formulations

- Why?
  - Compact objects with spin interesting for GW astronomy.
  - Canonical formalism efficient for calculating conservative dynamics.
  - Consistency check . . .
- Spinning test-bodies coupled to gravity:
  - H. P. Künzle, J. Math. Phys. **13**, 739 (1972).
  - K. Yee and M. Bander, PRD **48**, 2797 (1993).
  - E. Barausse, E. Racine, and A. Buonanno, PRD **80**, 104025 (2009).
- Dirac field coupled to gravity:
  - P. A. M. Dirac, in *Recent Developments in GR* (1962).
  - T. W. B. Kibble, J. Math. Phys. **4**, 1433 (1963).
  - S. Deser and C. J. Isham, PRD **14**, 2505 (1976).
  - J. Geheniau and M. Henneaux, GRG **8**, 611 (1977).
  - J. E. Nelson and C. Teitelboim, Ann. Phys. (N.Y.) **116**, 86 (1978).

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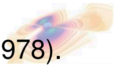
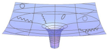
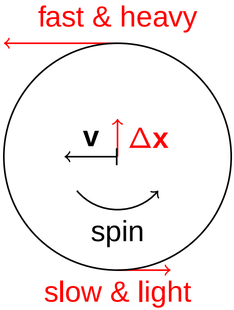




Spin in Special Relativity



- Spin is a 4-tensor $S^{\mu\nu}$:
  - Spin is $S^{ij} = \varepsilon^{ijk} S_k$.
  - Mass dipole related to S^{i0}.
- Different mass centers.
- Need spin supplementary condition:
  - Møller SSC: $\tilde{S}^{\mu 0} = 0$
  - Covariant SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{\mu 0} + \hat{S}^{\mu\nu} p_\nu = 0$

- In covariant SSC, with position **z**:

$$\{z^i, z^j\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}, \quad \dots$$

- In Newton-Wigner SSC:

$$\{\hat{z}^i, p_j\} = \delta_{ij}, \quad \{\hat{S}_i, \hat{S}_j\} = \varepsilon_{ijk} \hat{S}_k$$

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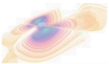

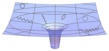

(3+1)-Decomposition

- Normal vector:

$$n_\mu = (-N, 0, 0, 0)\,, \quad n^\mu = \frac{1}{N}(1, -N^i)\,, \quad n_\mu n^\mu = -1$$

- Projector:

$$\gamma^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \quad g_{ij} = \gamma_{ij}\,, \quad \gamma_{ik}\gamma^{kj} = \delta_{ij}$$

- Extrinsic curvature:

$$K_{ij} \equiv -n_{(i||j)}$$

$$\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{lj} - \gamma^{ij}\gamma^{kl})K_{kl}$$

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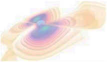
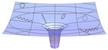




Point-Mass Action in ADM Form

$$W = \int dt\, p_i \dot{z}^i + \int d^4x \left[\frac{1}{16\pi} \pi^{ij} \gamma_{ij,0} - N\mathscr{H} + N^i \mathscr{H}_i + (\mathrm{st}) \right]$$

- Constraint equations:

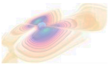
$$0 = \mathscr{H} \equiv -\frac{1}{16\pi\sqrt{\gamma}} \left[\gamma \mathrm{R} + \frac{1}{2} \left(\gamma_{ij} \pi^{ij} \right)^2 - \gamma_{ij} \gamma_{kl} \pi^{ik} \pi^{jl} \right] + \mathscr{H}^{\mathrm{matter}}$$

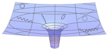
$$0 = \mathscr{H}_i \equiv \frac{1}{8\pi} \gamma_{ij} \pi^{jk}{}_{;k} + \mathscr{H}_i^{\mathrm{matter}}$$

- Source terms are related to the stress-energy tensor $T^{\mu\nu}$:

$$\mathscr{H}^{\mathrm{matter}} \equiv \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu = \sqrt{m^2 + \gamma^{ij} p_i p_j}\, \delta$$

$$\mathscr{H}_i^{\mathrm{matter}} \equiv -\sqrt{\gamma} T_{i\nu} n^\nu = p_i \delta$$





ADM Canonical Formalism

- Hamiltonian without gauge fixing:

$$H[z^i, p_i, \gamma_{ij}, \pi^{ij}] = \int d^3\mathbf{x}(N\mathscr{H} - N^i\mathscr{H}_i) + E[\gamma_{ij}]$$

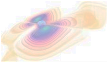
$$E[\gamma_{ij}] = \frac{1}{16\pi} \oint d^2 s_i(\gamma_{ij,j} - \gamma_{jj,i})$$

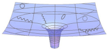
- ADM Hamiltonian (Hamiltonian in ADMTT gauge)
  $\hat{=}$ ADM Energy depending on canonical variables:

$$H_{\text{ADM}} = E[z^i, p_i, h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}] = -\frac{1}{16\pi}\int \mathrm{d}^3\mathbf{x}\, \Delta\phi$$

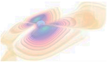
$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h^{\text{TT}}_{ij}, \quad \pi^{ii} = 0$$

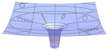
- Matter only Hamiltonian: Elimination of h^{TT}_{ij} and π^{ij}_{TT}.





Outline



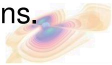


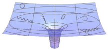
Non-Relativistic Spherical Top

- Body fixed coordinates $z^{[i]}$: $z^i(t) = R_{[j]i}(t) z^{[j]}$, $R_{[k]i} R_{[k]j} = \delta_{ij}$
- Independent angle variables: $R_{[i]j} = R_{[i]j}(\phi, \psi, \theta)$
- Angular velocity tensor: $\Omega^{ij} = \varepsilon_{ijk} \Omega^k = R_{[k]i} \dot{R}_{[k]j}$
- Lagrangian: $L = \frac{1}{4} J \Omega^{ij} \Omega^{ij}$
- Spin tensor: $S_{ij} = 2 \frac{\partial L}{\partial \Omega^{ij}} = J \Omega^{ij}$
- Legendre transformed:

$$L = \frac{1}{2} S_{ij} \Omega^{ij} - H[R_{ij}, S_{ij}] , \quad H = \frac{1}{4J} S_{ij} S_{ij}$$

- Trick: Use $\delta\theta_{ij} = -\delta\theta_{ji} = R_{[k]i} \delta R_{[k]j}$ as independent variations.
- Usual Poisson brackets.

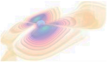


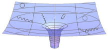


Relativistic Spherical Top

E.g., Hanson and Regge (1974)

- No rigid bodies.
- Mathematical abstraction: Top is
  - Worldline with Lorentz-matrix $\Lambda_{A\mu}$. $\quad \eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
  - $\Lambda_{A\mu}$ is pure rotation in rest-frame: $\Lambda_{A\mu} = \begin{pmatrix} -1 & 0 \\ 0 & R_{[i]j} \end{pmatrix}$
- Equivalent description: $\Lambda_{[0]\mu} = p_\mu / m \quad$ or $\quad \Lambda^{[i]\mu} p_\mu = 0$
- Angular velocity tensor: $\Omega^{\mu\nu} = \Lambda_A{}^{\mu} \dfrac{d\Lambda^{A\nu}}{d\tau}$
- Spin tensor: $S_{\mu\nu} = 2 \dfrac{\partial L}{\partial \Omega^{\mu\nu}}$
- Associated SSC: $S_{\mu\nu} p^\mu = 0$





Minimal Coupling

- Problem with metric variation: (also $\Lambda_{A\mu} = e_{A\mu}$)

$$\Lambda_{A\mu}\Lambda^{A}{}_{\nu} = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$

- Variate Λ^{Aa} and tetrad $e_{a\mu}$, $\Lambda^{A}{}_{\mu} = \Lambda^{Aa} e_{a\mu}$:

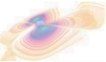
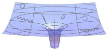
$$\Lambda_{Aa}\Lambda^{A}{}_{b} = \eta_{ab} \quad \leftrightarrow \quad \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

- Matter Lagrangian density and constraints:

$$\mathscr{L}_M = \int d\tau \left[p_\mu u^\mu + \frac{1}{2} S_{ab}\Omega^{ab} \right] \delta_{(4)}$$

$$\Omega^{ab} = \Lambda_A{}^a \frac{D\Lambda^{Ab}}{d\tau} = \Lambda_A{}^a \left[\frac{d\Lambda^{Ab}}{d\tau} - \Lambda^{A}{}_{c}\,\omega_\mu{}^{cb} u^\mu \right]$$

$$S_{ab}p^b = 0\,, \quad \Lambda^{[i]a}p_a = 0\,, \quad p_\mu p^\mu + m^2 = 0$$

Result of Variation

- Approximated linear in spin.
- Field equations with stress-energy tensor (Mathisson, Tulczyjew):

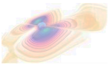
$$\sqrt{-g}\; T^{\mu\nu} = \int d\tau \left[m u^{\mu} u^{\nu} \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$

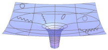
- EOM (Mathisson, Papapetrou):

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \qquad \frac{Dp_{\mu}}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^{\gamma} R_{\mu\gamma\nu\lambda}$$

- Original derivation: Evaluate $T^{\mu\nu}{}_{||\nu} = 0$ with ansatz

$$\sqrt{-g}\, T^{\mu\nu} = \int d\tau \left[t^{\mu\nu} \delta_{(4)} + (t^{\mu\nu\alpha} \delta_{(4)})_{||\alpha} \right]$$





Reduction of the Matter Part

- Solve matter constraints, Schwinger time gauge $e_{(0)\mu} = n_\mu$, $\tau = t$.
- Variable redefinitions:

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

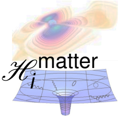
$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p_{(j)}}{m(m - np)} \right), \quad \gamma_{ik}\gamma_{jl} A^{kl} = \frac{1}{2}\hat{S}_{ij} + \frac{m p_{(i} nS_{j)}}{np(m - np)}$$

$$\hat{p}_i = p_i + K_{ij} nS^j + A^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np} \right) \Gamma^{kj}{}_i$$

- Matter Lagrangian density now, with $\hat{\Omega}^{(i)(j)} = \hat{\Lambda}_{[k]}{}^{(i)} \dot{\hat{\Lambda}}^{[k](j)}$,

$$\mathscr{L}_M = A^{ij} e_{(k)i} e^{(k)}_{j,0} \hat{\delta} + \hat{p}_i \dot{\hat{z}}^i \hat{\delta} + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} \hat{\delta} - N \mathscr{H}^{\text{matter}} + N^i \mathscr{H}_i^{\text{matter}}$$



ADM Formalism with Spin

- Legendre transformation for gravitational field.
- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = \gamma_{ik} \quad \Rightarrow \quad (e_{ij}) = \sqrt{(\gamma_{ij})}$$

- Action:

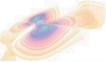
$$W = \frac{1}{16\pi} \int d^4 x \hat{\pi}^{ij} \gamma_{ij,0} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H \right]$$

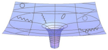
$$H = \int d^3 x (N \mathscr{H} - N^i \mathscr{H}_i) + E[\gamma_{ij}]$$

- Canonical field momentum:

$$\hat{\pi}^{ij} = \pi^{ij} + 8\pi A^{(ij)} \hat{\delta} + 16\pi B^{ij}_{kl} A^{[kl]} \hat{\delta}$$

$$e_{k[i} e_{j]k,0} = B^{kl}_{ij} \gamma_{kl,0}$$





Full Reduction

- Solve field constraints in ADMTT gauge.
- Fully reduced action:

$$W = \frac{1}{16\pi} \int d^4x \hat{\pi}^{ij}_{\mathrm{TT}} h^{\mathrm{TT}}_{ij,0} + \int dt \left[\hat{p}_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H_{\mathrm{ADM}} \right]$$

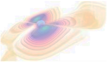
$$H_{\mathrm{ADM}} = E[\hat{z}^i, \hat{p}_i, \hat{S}_{(i)(j)}, h^{\mathrm{TT}}_{ij}, \hat{\pi}^{ij\mathrm{TT}}]$$

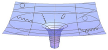
- Fundamental equal-time Poisson brackets:

$$\{\hat{z}^i, \hat{p}_j\} = \delta_{ij}\,, \quad \{\hat{S}_{(i)}, \hat{S}_{(j)}\} = \varepsilon_{ijk} \hat{S}_{(k)}$$

$$\{h^{\mathrm{TT}}_{ij}(\mathbf{x}, t), \hat{\pi}^{kl}_{\mathrm{TT}}(\mathbf{x}', t)\} = 16\pi \delta^{\mathrm{TT}kl}_{ij} \delta(\mathbf{x} - \mathbf{x}')$$

- Valid to all orders linear in spin.





Conserved Quantities

- Energy: $E = H_{\mathrm{ADM}}$
- Total linear and angular momentum:

$$P_i = \sum_a \hat{p}_{ai} - \frac{1}{16\pi} \int d^3x\, \hat{\pi}^{kl}_{\mathrm{TT}} h^{\mathrm{TT}}_{kl,i}$$

$$J_{ij} = \sum_a (\hat{z}^i_a \hat{p}_{aj} - \hat{z}^j_a \hat{p}_{ai}) + \sum_a \hat{S}_{a(i)(j)} - \frac{1}{16\pi} \int d^3x\, (x^i \hat{\pi}^{kl}_{\mathrm{TT}} h^{\mathrm{TT}}_{kl,j} - x^j \hat{\pi}^{kl}_{\mathrm{TT}} h^{\mathrm{TT}}_{kl,i}) - \frac{1}{16\pi} \int d^3x\, 2(\hat{\pi}^{ik}_{\mathrm{TT}} h^{\mathrm{TT}}_{kj} - \hat{\pi}^{jk}_{\mathrm{TT}} h^{\mathrm{TT}}_{ki})$$

- Boost: $J^{i0} \equiv K^i \equiv G^i - t\, P^i$
  With center-of-mass vector:

$$G^i = -\frac{1}{16\pi} \int \mathrm{d}^3\mathbf{x}\, x^i \Delta\phi$$

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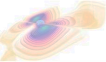
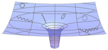




Stress-Energy-Tensor Algebra (Minkowski)

$$\{\mathcal{H}^{\mathrm{m}}(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\mathcal{H}_i^{\mathrm{m}}(x)\delta_{\mathbf{xx'},i} - \mathcal{H}_i^{\mathrm{m}}(x')\delta_{\mathbf{xx'},i}$$

$$\{\mathcal{H}_i^{\mathrm{m}}(x), \mathcal{H}^{\mathrm{m}}(x')\} = -\mathcal{H}^{\mathrm{m}}(x)\,\delta_{\mathbf{xx'},i} - T_{ij}(x')\,\delta_{\mathbf{xx'},j}$$

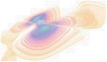
$$\{\mathcal{H}_i^{\mathrm{m}}(x), \mathcal{H}_j^{\mathrm{m}}(x')\} = -\mathcal{H}_j^{\mathrm{m}}(x)\,\delta_{\mathbf{xx'},i} - \mathcal{H}_i^{\mathrm{m}}(x')\,\delta_{\mathbf{xx'},j} + \partial_n \partial'_q \left[h_{injq}(x)\,\delta_{\mathbf{xx'}}\right]$$

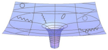
$$h_{injq}(x) = \left[-\hat{S}_{q)(n}\mathcal{P}_{i)(j} - \delta^{kl}\frac{\hat{p}_k \hat{S}_{l(n}\mathcal{P}_{i)(j}\hat{p}_{q)}}{n\hat{p}(m - n\hat{p})} + \delta^{kl}\frac{\hat{p}_k \hat{S}_{l(q}\mathcal{P}_{j)(i}\hat{p}_{n)}}{n\hat{p}(m - n\hat{p})} \right]\hat{\delta}$$

$$\mathcal{P}_{ij} \equiv \delta_{ij} - \frac{\hat{p}_i \hat{p}_j}{n\hat{p}^2}$$

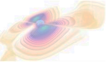
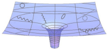
$$\mathcal{H}^{\mathrm{m}}(x) = T^{00}\,, \quad \mathcal{H}_i^{\mathrm{m}}(x) = T^{0i}$$

- Local version of the Poincaré algerbra.
- Minkowski limit of the gravitational constraint algebra.
- Dirac field also has $h_{injq}(x) \neq 0$.





Outline

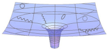
Order-by-Order Construction

- Hamiltonian $\hat{=}$ energy depending on canonical variables.
- Need to find canonical variables!
- Symmetries of the action:

$$P_i = \sum_a \hat{p}_{ai} + P_i^{\text{field}}, \quad J_i = \sum_a \left[\varepsilon_{ijk} \hat{z}_a^j \hat{p}_{ak} + \hat{S}_{a(i)} \right] + J_i^{\text{field}}$$

- Global Poincaré algebra:

$$\begin{aligned}
\{P_i, P_j\} &= 0\,, & \{P_i, H\} &= 0\,, & \{J_i, H\} &= 0 \\
\{J_i, P_j\} &= \varepsilon_{ijk} P_k\,, & \{J_i, J_j\} &= \varepsilon_{ijk} J_k\,, & \{J_i, G_j\} &= \varepsilon_{ijk} G_k \\
\{G_i, P_j\} &= H \delta_{ij}\,, & \{G_i, H\} &= P_i\,, & \{G_i, G_j\} &= -\varepsilon_{ijk} J_k
\end{aligned}$$



Canonical Variables at 2PN

- Calculate $\mathscr{H}_i^{\text{matter}}$:

$$\mathscr{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^{\nu}$$

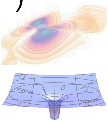
- Define canonical momentum $\hat{p}_i$ as:

$$\hat{p}_i = \int d^3\mathbf{x}\, \mathscr{H}_i^{\text{matter}}$$

- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} \hat{S}_{(m)}$ such that $\hat{\mathbf{S}}^2 = \text{const.}$
- $\hat{\mathbf{z}}$ and $e_{i(k)}$ fixed by

$$J_{ij} = \hat{z}^i \hat{p}_j - \hat{z}^j \hat{p}_i + \varepsilon_{ijm} \hat{S}_{(m)} = \int d^3\mathbf{x} \left(x^i \mathscr{H}_j^{\text{matter}} - x^j \mathscr{H}_i^{\text{matter}} \right)$$

since 1558



Outline

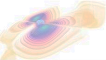
since 1558

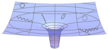
Next-to-Leading Order (NLO) Spin-Orbit

First derived: Damour, Jaranowski, and Schäfer (2008)

$$
\begin{aligned}
H_{SO}^{NLO} = &-\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{5m_2\hat{\mathbf{p}}_1^2}{8m_1^3} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{4m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4m_1m_2}\right. \\
&\left. + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{4m_1^2} + \frac{3(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2}{2m_1m_2}\right] \\
&+ \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{m_1m_2} + \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2}\right] \\
&+ \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{p}}_2)}{\hat{r}_{12}^2}\left[\frac{2(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2} - \frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})}{4m_1^2}\right] \\
&- \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[\frac{11m_2}{2} + \frac{5m_2^2}{m_1}\right] \\
&+ \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[6m_1 + \frac{15m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

since 1558

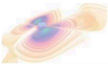


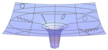


NLO Spin$_1$-Spin$_2$

Partial result: Porto and Rothstein (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1 S_2} = {} & \frac{1}{2 m_1 m_2 \hat{r}_{12}^3} \Big[\tfrac{3}{2} ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) + \tfrac{1}{2} (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \\
& + 6 ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) - \tfrac{1}{2} (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) \\
& - 15 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2) \\
& - 3 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) + 3 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3 (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + 3 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3 (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) - 3 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
& + \frac{3}{2 m_1^2 \hat{r}_{12}^3} \Big[-((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) \\
& + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
& + \frac{3}{2 m_2^2 \hat{r}_{12}^3} \Big[-((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) ((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) \\
& + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
& + \frac{6 (m_1 + m_2)}{\hat{r}_{12}^4} \Big[(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - 2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big]
\end{aligned}
$$



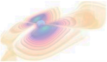


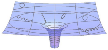
NLO Center of Mass

$$\mathbf{G}^{\text{NLO}}_{\text{SO}} = -\sum_a \frac{\hat{\mathbf{p}}_a^2}{8m_a^3}(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) + \sum_a \sum_{b \neq a} \frac{m_b}{4m_a \hat{r}_{ab}} \left[((\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{5\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} - 5(\hat{\mathbf{p}}_a \times \hat{\mathbf{S}}_a) \right] + \sum_a \sum_{b \neq a} \frac{1}{\hat{r}_{ab}} \left[\frac{3}{2}(\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) - \frac{1}{2}(\hat{\mathbf{n}}_{ab} \times \hat{\mathbf{S}}_a)(\hat{\mathbf{p}}_b \cdot \hat{\mathbf{n}}_{ab}) - ((\hat{\mathbf{p}}_b \times \hat{\mathbf{S}}_a) \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{z}}_a + \hat{\mathbf{z}}_b}{\hat{r}_{ab}} \right]$$

$$\mathbf{G}^{\text{NLO}}_{\text{S}_1\text{S}_2} = \frac{1}{2} \sum_a \sum_{b \neq a} \left\{ \left[3(\hat{\mathbf{S}}_a \cdot \hat{\mathbf{n}}_{ab})(\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) - (\hat{\mathbf{S}}_a \cdot \hat{\mathbf{S}}_b) \right] \frac{\hat{\mathbf{z}}_a}{\hat{r}_{ab}^3} + (\hat{\mathbf{S}}_b \cdot \hat{\mathbf{n}}_{ab}) \frac{\hat{\mathbf{S}}_a}{\hat{r}_{ab}^2} \right\}$$

$\Rightarrow$ Poincaré algebra is fulfilled.





Outline

The Stress-Energy Tensor with Quadrupole

- Ansatz:

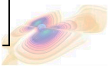
$$\sqrt{-g}\,T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

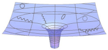
- Evaluate $T^{\mu\nu}{}_{||\nu} = 0$:   (with D. Puetzfeld)

$$\frac{D(S^{\mu\nu})}{D\tau} = 2p^{[\mu}u^{\nu]} + \frac{4}{3}\mathrm{R}^{[\mu}{}_{\rho\beta\alpha}J^{\nu]\rho\beta\alpha}$$

$$\frac{Dp_\mu}{D\tau} = -\frac{1}{2}\mathrm{R}_{\mu\rho\beta\alpha}u^\rho S^{\beta\alpha} - \frac{1}{6}\mathrm{R}_{\nu\rho\beta\alpha;\mu}J^{\nu\rho\beta\alpha}$$

$$\sqrt{-g}\,T^{\mu\nu} = \int d\tau \Big[u^{(\mu}p^{\nu)}\delta_{(4)} + \frac{1}{3}\mathrm{R}^{(\mu}{}_{\rho\beta\alpha}J^{\nu)\rho\beta\alpha}\delta_{(4)} + (u^{(\mu}S^{\nu)\alpha}\delta_{(4)})_{||\alpha} - \frac{2}{3}(J^{\mu\alpha\beta\nu}\delta_{(4)})_{||(\alpha\beta)} \Big]$$

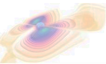


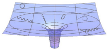


- Mass quadrupole $I^{\mu\nu}$: $J^{\nu\rho\beta\alpha} = -3u^{[\nu}I^{\rho][\beta}u^{\alpha]}$
- Spin-squared ansatz for $I^{\mu\nu}$. $\Rightarrow C_Q$ $(= 1$ for black holes$)$

NLO Spin_1-Spin_1 for Black Holes and Neutron Stars

$$
\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{S}_1^2} = \frac{m_2}{m_1^3 \hat{r}_{12}^3} \Bigg[& \left(\frac{15}{4} - \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(\frac{5}{4} - \frac{5}{4} C_Q\right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \\
& + \left(-\frac{9}{8} + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 + \left(-\frac{21}{8} + \frac{9}{4} C_Q\right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(-\frac{5}{4} + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 \Bigg] + \frac{C_Q}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] \\
+ \frac{1}{m_1^2 \hat{r}_{12}^3} \Bigg[& \left(-\frac{3}{2} + \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) \\
& + \left(-3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(-\frac{3}{2} + \frac{9}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\
& + \left(\frac{3}{2} - \frac{3}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 + \left(3 - \frac{21}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - \frac{15}{4} C_Q (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \Bigg] \\
+ \frac{m_2}{\hat{r}_{12}^4} \Bigg[& \left(2 + \frac{1}{2} C_Q\right) \hat{\mathbf{S}}_1^2 - \left(3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg] \\
+ \frac{m_2^2}{m_1 \hat{r}_{12}^4} \Bigg[& (1 + 2 C_Q) \hat{\mathbf{S}}_1^2 - (1 + 6 C_Q)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg]
\end{aligned}
$$





Thank you for your attention

and the German Research Foundation DFG for support

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