Canonical Formulation of Spin in General Relativity

Jan Steinhoff   Steven Hergt   Gerhard Schäfer

since 1558

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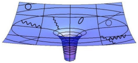
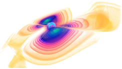




DFG : GRK 1523 “Quantum and Gravitational Fields” and SFB/TR7

ADM Formalism and PN Approximation

ADM stands for Arnowitt, Deser, Misner; PN stands for post-Newton

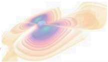
- $H^{\text{ADM}} \widehat{=}$ ADM energy expressed in terms of canonical variables after field constraints are solved in the ADMTT gauge.
- Canonical Matter variables enter through source terms of the field constraints, e.g., for a point-mass:

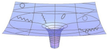
$$\mathcal{H}^{\text{matter}} = \sqrt{m^2 + \gamma^{ij} p_i p_j}\,\delta\,, \quad \mathcal{H}_i^{\text{matter}} = p_i \delta\,, \quad \delta \equiv \delta(x^i - z^i)$$

- These source terms follow from the energy-momentum tensor $T^{\mu\nu}$.
- H^{ADM} can not be written down explicitly.
- Approximations are possible, e.g., post-Newton:

$$H_{\text{N}}^{\text{ADM}} = \int \mathrm{d}^3 x \left[\underbrace{\mathcal{H}_{(4)}^{\text{matter}}}_{\text{kinetic energy density}} - \underbrace{\frac{1}{8}\phi_{(2)}}_{\text{Newtonian potential}} \underbrace{\mathcal{H}_{(2)}^{\text{matter}}}_{\text{mass density}} \right]$$

$H_{\text{1PN}}^{\text{ADM}} =$ integrals over δ $\Rightarrow$ „simple“





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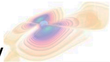
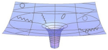
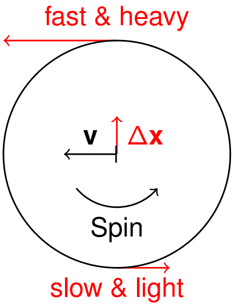




Spin in Special Relativity

- Components of 4-Spin $S^{\mu\nu} = -S^{\nu\mu}$:
  - 3-Spin $S^{ij} = \epsilon^{ijk} S_k$
  - Mass dipole S^{i0}
- Center-of-mass is frame-dependent.
- Need spin supplementary condition (SSC):
  - Møller SSC: $\tilde{S}^{\mu 0} = 0$
  - Covariant SSC: $S^{\mu\nu} p_\nu = 0$
  - Newton-Wigner (canonical) SSC: $m\hat{S}^{\mu 0} + \hat{S}^{\mu\nu} p_\nu = 0$



Canonical structure

In covariant SSC, with associated center **z**:

$$\{z^i, z^j\} = \frac{S^{ij}}{m^2} - \frac{p^i S^{0j} - p^j S^{0i}}{m^2 p^0}, \quad \dots$$

In Newton-Wigner SSC:

$$\{\hat{z}^i, p_j\} = \delta_{ij}, \quad \{\hat{S}_i, \hat{S}_j\} = \epsilon_{ijk} \hat{S}_k$$

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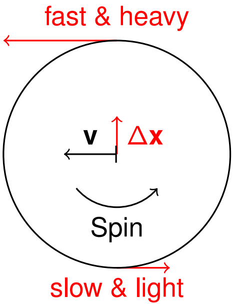

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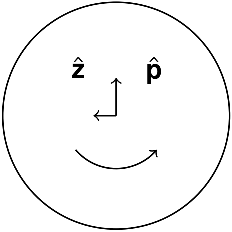

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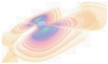
Angular Velocity and Spin

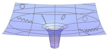
in Newtonian mechanics and special relativity

	Newton	special relativity
body-fixed frame	$x^{[i]} = \Lambda_{[i]j} x^j$	
rotational degrees of freedom	$\Lambda_{[k]i}\Lambda_{[k]j} = \delta_{ij}$	$\eta^{AB}\Lambda_{A\mu}\Lambda_{B\nu} = \eta_{\mu\nu}$
$\hookrightarrow$ supplementary condition		$\Lambda^{[i]\mu} p_\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda_{[k]i} \frac{\mathrm{d}\Lambda_{[k]j}}{\mathrm{d}t}$	$\Omega^{\mu\nu} = \Lambda_A{}^{\mu} \frac{\mathrm{d}\Lambda^{A\nu}}{\mathrm{d}\tau}$
Spin (L: Lagrangian)	$S_{ij} = 2\frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2\frac{\partial L}{\partial \Omega^{\mu\nu}}$
$\hookrightarrow$ supplementary condition		$S_{\mu\nu} p^\nu = 0$

Remark:

- Angular velocity vector is $\Omega^i = \frac{1}{2}\epsilon_{ijk}\Omega^{jk}$. Analogous for spin.





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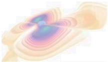
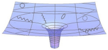




Action Approach with Minimal Coupling

- Metric variation problematic:

$$\Lambda_{A\mu}\Lambda^{A}{}_{\nu} = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$$

- Variate Λ^{Aa} and tetrad $e_{a\mu}$, $e_{a\mu}e^{a}{}_{\nu} = g_{\mu\nu}$, $\Lambda^{A}{}_{\mu} = \Lambda^{Aa}e_{a\mu}$:

$$\Lambda_{Aa}\Lambda^{A}{}_{b} = \eta_{ab} \quad \leftrightarrow \quad \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$

- Minimal coupling:

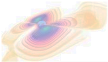
$$e^{a}{}_{\mu}e^{b}{}_{\nu}\Omega^{\mu\nu} = \Omega^{ab} = \Lambda_{A}{}^{a}\frac{D\Lambda^{Ab}}{d\tau} = \Lambda_{A}{}^{a}\left[\frac{d\Lambda^{Ab}}{d\tau} - \Lambda^{A}{}_{c}\omega_{\mu}{}^{cb}u^{\mu}\right]$$

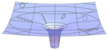
- Supplementary conditions and mass-shell constraint:

$$S_{\mu\nu}p^{\nu} = 0\,, \quad \Lambda^{[i]a}e_{a\nu}p^{\nu} = 0\,, \quad p_\mu p^\mu + m^2 = 0$$

- Solve constraints, supplementary and gauge conditions.
- Find variables, in which Lagrangian is of the canonical form

$$L = p_i\dot{q}^i - H$$





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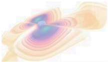

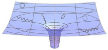

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$$L = \frac{1}{16\pi}\int \mathrm{d}^3x\, \hat{\pi}^{ij\mathrm{TT}}\hat{h}^{\mathrm{TT}}_{ij,0} + \hat{p}_i\dot{\hat{z}}^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Omega}^{(i)(j)} - H^{\mathrm{ADM}}$$

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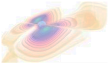

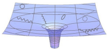

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Canonical structure

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \{A, H^{\mathrm{ADM}}\} + \frac{\partial A}{\partial t}$$

$$\{\hat{z}^i, \hat{p}_j\} = \delta_{ij}$$

$$\{\hat{\Lambda}^{[i](j)}, \hat{S}_{(k)(l)}\} = \hat{\Lambda}^{[i](k)}\delta_{lj} - \hat{\Lambda}^{[i](l)}\delta_{kj}$$

$$\{\hat{S}_{(i)(j)}, \hat{S}_{(k)(l)}\} = \delta_{ik}\hat{S}_{(j)(l)} - \delta_{jk}\hat{S}_{(i)(l)} - \delta_{il}\hat{S}_{(j)(k)} + \delta_{jl}\hat{S}_{(i)(k)}$$

$$\{\hat{h}^{\mathrm{TT}}_{ij}(\mathbf{x}), \hat{\pi}^{kl\mathrm{TT}}(\mathbf{x}')\} = 16\pi\delta^{\mathrm{TT}kl}_{ij}\delta(\mathbf{x} - \mathbf{x}')$$

$$\delta^{\mathrm{TT}ij}_{kl} \equiv \text{TT-projector}$$

- Variate Λ^{A}
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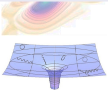


Canonical Variables to Linear Order in Spin

- Gauges: $e_{(0)\mu} = n_\mu$, $(e_{(i)j}) = \sqrt{(\gamma_{ij})}$, $\tau = t$
- Matter variables: $\hat{z}^i$ and $\hat{S}_{ij}$ from SSC $\hat{S}^{\mu\nu}(p_\nu + m n_\nu) = 0$

$$z^i = \hat{z}^i - \frac{nS^i}{m - np}, \quad np = -\sqrt{m^2 + \gamma^{ij} p_i p_j}$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i nS_j}{m - np} + \frac{p_j nS_i}{m - np}, \quad nS_i = -\frac{p_k \gamma^{kj} \hat{S}_{ji}}{m}$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p_{(j)}}{m(m - np)} \right), \quad \gamma_{ik}\gamma_{jl} A^{kl} = \frac{1}{2}\hat{S}_{ij} + \frac{m p_{(i} nS_{j)}}{np(m - np)}$$

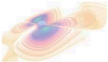
$$p_i = \hat{p}_i - K_{ij} nS^j - A^{kl} e_{(j)k} e^{(j)}{}_{l,i} + \left(\frac{1}{2} S_{kj} + \frac{p_{(k} nS_{j)}}{np} \right) \Gamma^{kj}{}_i$$

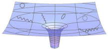
- Field variables:

$$h^{\mathrm{TT}}_{ij} = \hat{h}^{\mathrm{TT}}_{ij}$$

$$\pi^{ij\mathrm{TT}} = \hat{\pi}^{ij\mathrm{TT}} - \delta^{\mathrm{TT}ij}_{kl} \left(8\pi A^{(kl)} \delta + 16\pi B^{kl}_{mn} A^{[mn]} \delta \right)$$

$$2B^{kl}_{mn} \equiv e^{(i)}{}_m \frac{\partial e_{(i)n}}{\partial \gamma_{kl}} - e^{(i)}{}_n \frac{\partial e_{(i)m}}{\partial \gamma_{kl}}, \quad \delta^{\mathrm{TT}ij}_{kl} \equiv \text{TT-projector}$$

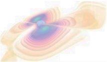


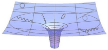


Next-to-Leading Order (NLO) Spin-Orbit

Hamiltonian first derived: Damour, Jaranowski, Schäfer (2008)
See also: Tagoshi, Ohashi, Owen (2001); Faye, Blanchet, Buonanno (2006)

$$
\begin{aligned}
H^{\text{NLO}}_{\text{SO}} = & -\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{5 m_2 \hat{\mathbf{p}}_1^2}{8 m_1^3} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{4 m_1^2} - \frac{3\hat{\mathbf{p}}_2^2}{4 m_1 m_2} \right. \\
& \left. + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{4 m_1^2} + \frac{3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2}{2 m_1 m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)}{m_1 m_2} + \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2}\right] \\
& + \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot \hat{\mathbf{p}}_2)}{\hat{r}_{12}^2}\left[\frac{2(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})}{m_1 m_2} - \frac{3(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})}{4 m_1^2}\right] \\
& - \frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
& + \frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot \hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[6 m_1 + \frac{15 m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$





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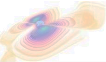
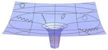
H_{\mathrm{SO}}^{\mathrm{NLO}} = &-\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{5m_2\hat{\mathbf{p}}_1^2}{8m_1^3}+\frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{4m_1^2}-\frac{3\hat{\mathbf{p}}_2^2}{4m_1m_2}\right.\\
&\left.+\frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{4m_1^2}+\frac{3(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2}{2m_1m_2}\right]\\
&+\frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^2}\left[\frac{(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2)}{m_1m_2}+\frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2}\right]\\
&+\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{p}}_2)}{\hat{r}_{12}^2}\left[\frac{2(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})}{m_1m_2}-\frac{3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})}{4m_1^2}\right]\\
&-\frac{((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[\frac{11m_2}{2}+\frac{5m_2^2}{m_1}\right]\\
&+\frac{((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})}{\hat{r}_{12}^3}\left[6m_1+\frac{15m_2}{2}\right]+(1\leftrightarrow 2)
\end{aligned}
$$

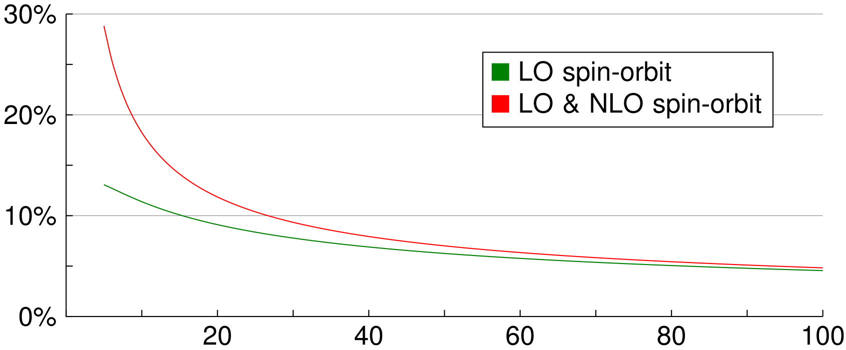
for circular orbits

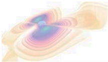


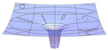


Perihelion Advance with NLO Spin-Orbit

- Y-axis: deviation of perihelion advance relative to non-spinning case
- X-axis: distance in units of the total mass
- Parameters: eccentricity 0.1, $m_1 = m_2$, $S_1/m_1^2 = 0.8$







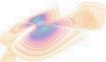
- For formulas see Tessmer, Hartung, Schäfer,
  Class. Quant. Grav. 27, 165005, (2010), arXiv:1003.2735 [gr-qc].
- Thanks to Johannes Hartung for delivering this plot.

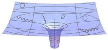
NLO Spin$_1$-Spin$_2$

Partial result: Porto, Rothstein (2006). Full result: Steinhoff, Hergt, Schäfer (2008).

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1S_2} = {} & \frac{1}{2m_1m_2\hat{r}_{12}^3}\Big[\tfrac{3}{2}((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12}) + \tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2) \\
& + 6((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12}) - \tfrac{1}{2}(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1) \\
& - 15(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2) \\
& - 3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{p}}_2) + 3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}) + 3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12}) - 3(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big] \\
& + \frac{3}{2m_1^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12}) \\
& \quad + (\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{n}}_{12})\Big] \\
& + \frac{3}{2m_2^2\hat{r}_{12}^3}\Big[-((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_2)\cdot\hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2\times\hat{\mathbf{S}}_1)\cdot\hat{\mathbf{n}}_{12}) \\
& \quad + (\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2\cdot\hat{\mathbf{n}}_{12})\Big] \\
& + \frac{6(m_1+m_2)}{\hat{r}_{12}^4}\Big[(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{S}}_2) - 2(\hat{\mathbf{S}}_1\cdot\hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2\cdot\hat{\mathbf{n}}_{12})\Big]
\end{aligned}
$$

since 1558



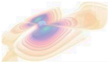


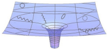
NLO Spin$_1$-Spin$_2$

Partial result: Porto, Rothstein (2006). Full result: Steinhoff, Hergt, Schäfer (2008).

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1 S_2} = \frac{1}{2 m_1 m_2 \hat{r}_{12}^3} \Big[& \tfrac{3}{2}((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) + \tfrac{1}{2}(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \\
& + 6((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) - \tfrac{1}{2}(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1) \\
& - 15(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2) \\
& - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) + 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \\
& + 3(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) - 3(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
+ \frac{3}{2 m_1^2 \hat{r}_{12}^3} \Big[& -((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_1 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12}) \\
& + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
+ \frac{3}{2 m_2^2 \hat{r}_{12}^3} \Big[& -((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_2) \cdot \hat{\mathbf{n}}_{12})((\hat{\mathbf{p}}_2 \times \hat{\mathbf{S}}_1) \cdot \hat{\mathbf{n}}_{12}) \\
& + (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})^2 - (\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big] \\
+ \frac{6(m_1 + m_2)}{\hat{r}_{12}^4} \Big[& (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2) - 2(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_2 \cdot \hat{\mathbf{n}}_{12}) \Big]
\end{aligned}
$$

since 1558





for circular orbits and orbital-angular-momentum-aligned spins

Quadrupole Deformation due to Spin

- Quadratic order in spin $\rightarrow$ quadrupole deformation
- Ansatz for Dixon's quadrupole:

$$J^{\nu\rho\beta\alpha} = -3 u^{[\nu} Q^{\rho][\beta} u^{\alpha]} \,, \qquad Q_{\mu\nu} = \frac{C_Q}{m_p} S_{\mu\rho} S_{\nu}{}^{\rho} - \text{Trace}$$

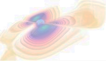
- C_Q is an object-dependent constant:
  - Neutron stars: $C_Q = 4 \ldots 8$ [Poisson (1998); Laarakkers, Poisson (1999)]
  - Black holes: $C_Q = 1$

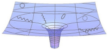
- Approach via effective action possible [cf. Porto, Rothstein (2008)]:

$$L_{S^2} = \underbrace{-\frac{1}{2m} \mathrm{R}_{\mu\nu\alpha\beta} S^{\rho\mu} S^{\alpha\beta} \frac{u^{\nu} u_{\rho}}{\sqrt{-u_{\sigma} u^{\sigma}}}}_{\text{preserves supplementary conditions}} \underbrace{-\frac{1}{2} \mathrm{R}_{\alpha\mu\beta\nu} Q^{\alpha\beta} \frac{u^{\mu} u^{\nu}}{\sqrt{-u_{\sigma} u^{\sigma}}}}_{\text{quadrupole deformation}}$$

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L_{S^2}}{\partial \mathrm{R}_{\mu\nu\alpha\beta}} \qquad \text{[Bailey, Israel (1975)]}$$

- $K_{ij,0}$ in matter action problematic for canonical formulation.





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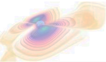
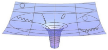




- $K_{ij,0}$ in matter action problematic for canonical formulation.

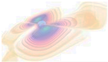
Shortcut to NLO Spin(1)-Spin(1) Hamiltonian

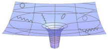
- $\hat{p}_i$-dependent part of $H^{\text{NLO}}_{S_1^2}$ follows from the global Poincaré algebra.
  [Hergt, Schäfer (2008)]
- Only need $\hat{p}_i = 0$ part of $H^{\text{NLO}}_{S_1^2}$.
- Only need $\hat{p}_i = 0$ part of matter energy density.
- $\hat{p}_i = 0$ part of matter energy density can be calculated from

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \Big[u^{(\mu} p^{\nu)} \delta_{(4)} + \left(u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} + \frac{1}{3} \text{R}_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left(J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} \Big]$$

[Steinhoff, Pützfeld (2010)]

$$J^{\nu\rho\beta\alpha} = -3 u^{[\nu} Q^{\rho][\beta} u^{\alpha]} \,, \quad Q_{\mu\nu} = \frac{C_Q}{m_p} S_{\mu\rho} S_\nu{}^\rho - \text{Trace}$$



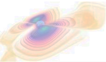


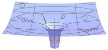
NLO Spin$_1$-Spin$_1$ for Black Holes and Neutron Stars

For black holes: Steinhoff, Hergt, Schäfer (2008). See also: Porto, Rothstein (2008).

$$
\begin{aligned}
H^{\rm NLO}_{S_1^2} = \frac{m_2}{m_1^3 \hat{r}_{12}^3} \Bigg[& \left(\frac{15}{4} - \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(\frac{5}{4} - \frac{5}{4} C_Q\right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \\
& + \left(-\frac{9}{8} + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 + \left(-\frac{21}{8} + \frac{9}{4} C_Q\right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(-\frac{5}{4} + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 \Bigg] + \frac{C_Q}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] \\
+ \frac{1}{m_1^2 \hat{r}_{12}^3} \Bigg[& \left(-\frac{3}{2} + \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) \\
& + \left(-3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(-\frac{3}{2} + \frac{9}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\
& + \left(\frac{3}{2} - \frac{3}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 + \left(3 - \frac{21}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - \frac{15}{4} C_Q (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \Bigg] \\
+ \frac{m_2}{\hat{r}_{12}^4} \Bigg[& \left(2 + \frac{1}{2} C_Q\right) \hat{\mathbf{S}}_1^2 - \left(3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg] \\
+ \frac{m_2^2}{m_1 \hat{r}_{12}^4} \Bigg[& (1 + 2 C_Q) \hat{\mathbf{S}}_1^2 - (1 + 6 C_Q)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg]
\end{aligned}
$$

since 1558





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\begin{aligned}
H^{\mathrm{NLO}}_{S_1^2} = \frac{m_2}{m_1^3 \hat{r}_{12}^3} \Bigg[& \left(\frac{15}{4} - \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) + \left(\frac{5}{4} - \frac{5}{4} C_Q\right) \hat{\mathbf{p}}_1^2 \hat{\mathbf{S}}_1^2 \\
& + \left(-\frac{9}{8} + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \hat{\mathbf{S}}_1^2 + \left(-\frac{21}{8} + \frac{9}{4} C_Q\right) \hat{\mathbf{p}}_1^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& + \left(-\frac{5}{4} + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)^2 \Bigg] + \frac{C_Q}{m_1 m_2 \hat{r}_{12}^3} \left[\frac{9}{4} \hat{\mathbf{p}}_2^2 (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 - \frac{3}{4} \hat{\mathbf{p}}_2^2 \hat{\mathbf{S}}_1^2 \right] \\
+ \frac{1}{m_1^2 \hat{r}_{12}^3} \Bigg[& \left(-\frac{3}{2} + \frac{9}{2} C_Q\right) (\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1) \\
& + \left(-3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) + \left(-\frac{3}{2} + \frac{9}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2) \hat{\mathbf{S}}_1^2 \\
& + \left(\frac{3}{2} - \frac{3}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12}) \hat{\mathbf{S}}_1^2 + \left(3 - \frac{21}{4} C_Q\right) (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \\
& - \frac{15}{4} C_Q (\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{n}}_{12})(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 + \left(\frac{3}{2} - \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{p}}_2) \Bigg] \\
+ \frac{m_2}{\hat{r}_{12}^4} \Bigg[& \left(2 + \frac{1}{2} C_Q\right) \hat{\mathbf{S}}_1^2 - \left(3 + \frac{3}{2} C_Q\right) (\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg] \\
+ \frac{m_2^2}{m_1 \hat{r}_{12}^4} \Bigg[& (1 + 2 C_Q) \hat{\mathbf{S}}_1^2 - (1 + 6 C_Q)(\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{n}}_{12})^2 \Bigg]
\end{aligned}
$$

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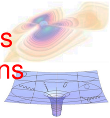
for circular orbits and aligned spins



Thank you for your attention

and the German Research Foundation DFG for support

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