ADM canonical formulation with spin

and application to post-Newtonian approximations

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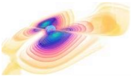
12th Marcel Grossmann Meeting, 2009, Paris



DFG: SFB/TR7 Gravitational Wave Astronomy

ADM Hamiltonian and Global Poincaré Invariance

- Hamiltonian in ADMTT gauge (ADM Hamiltonian)
  $\widehat{=}$ ADM energy depending on canonical variables:

$$H_{\mathrm{ADM}} = E[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij}] = -\frac{1}{16\pi} \int \mathrm{d}^3\mathbf{x}\, \Delta\phi$$

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

- The algebra of global Poincaré invariance reads

$$\{P_i, P_j\} = 0\,, \qquad \{P_i, H\} = 0\,, \qquad \{J_i, H\} = 0\,,$$

$$\{J_i, P_j\} = \varepsilon_{ijk} P_k\,, \qquad \{J_i, J_j\} = \varepsilon_{ijk} J_k\,, \qquad \{J_i, G_j\} = \varepsilon_{ijk} G_k\,,$$

$$\{G_i, P_j\} = H\delta_{ij}\,, \qquad \{G_i, H\} = P_i\,, \qquad \{G_i, G_j\} = -\varepsilon_{ijk} J_k\,,$$

with

$$P_i = \sum_a p_{ai}\,, \quad J_i = \sum_a \left[\varepsilon_{ijk} \hat{z}_a^j p_{ak} + S_{a(i)}\right].$$

Spin in GR

- Stress-energy tensor density in covariant SSC, $S^{\mu\nu}u_\nu = 0$:

$$\sqrt{-g}\;T^{\mu\nu} = \int d\tau \left[m u^\mu u^\nu \delta_{(4)} - (S^{\alpha(\mu} u^{\nu)} \delta_{(4)})_{||\alpha} \right]$$

$$\delta_{(4)} \equiv \delta(x - z(\tau))$$

- EOM follow from $T^{\mu\nu}{}_{||\nu} = 0$:

$$\frac{DS^{\mu\nu}}{d\tau} = 0\,, \quad m\frac{Du_\mu}{d\tau} = \frac{1}{2} S^{\lambda\nu} u^\gamma R^{(4)}_{\mu\gamma\nu\lambda}$$

Identification of Canonical Variables

- Calculate $\mathscr{H}_i^{\text{matter}}$:

$$\mathscr{H}_i^{\text{matter}} = -\sqrt{\gamma} T_{i\nu} n^\nu$$

- Define canonical momentum p_i as:

$$p_i = \int d^3\mathbf{x}\, \mathscr{H}_i^{\text{matter}}$$

- Define spin $\hat{S}_{ij} = e_{i(k)} e_{j(l)} \varepsilon_{klm} S_{(m)}$ such that $\mathbf{S}^2 = \text{const.}$ and

$$J_{ij} = \hat{z}^i p_j - \hat{z}^j p_i + \varepsilon_{ijm} S_{(m)} = \int d^3\mathbf{x}\, (x^i \mathscr{H}_j^{\text{matter}} - x^j \mathscr{H}_i^{\text{matter}})$$

- Go over to canonical position variable **z** by a Lie shift (such that one has the Newton-Wigner SSC in flat space).

NLO Spin-Orbit Hamiltonian

First derived: Damour, Jaranowski, and Schäfer (2008)

$$
\begin{aligned}
H_{SO}^{NLO} = &-\frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^2}\left[\frac{5 m_2 \mathbf{p}_1^2}{8 m_1^3} + \frac{3(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^2} - \frac{3\mathbf{p}_2^2}{4 m_1 m_2}\right. \\
&\left. + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{4 m_1^2} + \frac{3(\mathbf{p}_2\cdot\mathbf{n}_{12})^2}{2 m_1 m_2}\right] \\
&+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^2}\left[\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1 m_2}\right] \\
&+ \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{p}_2)}{r_{12}^2}\left[\frac{2(\mathbf{p}_2\cdot\mathbf{n}_{12})}{m_1 m_2} - \frac{3(\mathbf{p}_1\cdot\mathbf{n}_{12})}{4 m_1^2}\right] \\
&- \frac{((\mathbf{p}_1 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^3}\left[\frac{11 m_2}{2} + \frac{5 m_2^2}{m_1}\right] \\
&+ \frac{((\mathbf{p}_2 \times \mathbf{S}_1)\cdot\mathbf{n}_{12})}{r_{12}^3}\left[6 m_1 + \frac{15 m_2}{2}\right] + (1 \leftrightarrow 2)
\end{aligned}
$$

- Equivalent to Faye, Blanchet, Buonanno (2007)

NLO Spin$_1$-Spin$_2$ Hamiltonian

Partial result: Porto and Rothstein (2006)

$$
\begin{aligned}
H^{\mathrm{NLO}}_{S_1S_2} = \frac{1}{2m_1m_2r_{12}^3}\Big[& \tfrac{3}{2}((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2) \\
& + 6((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) - \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) \\
& - 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) \\
& - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\Big] \\
+ \frac{3}{2m_1^2r_{12}^3}\Big[& -((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})\Big] \\
+ \frac{3}{2m_2^2r_{12}^3}\Big[& -((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) \\
& + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})\Big] \\
+ \frac{6(m_1+m_2)}{r_{12}^4}\Big[& (\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})\Big]
\end{aligned}
$$

Hamiltonians from the Poincaré Algebra

Hergt and Schäfer (2008)

The full Hamiltonian up to 2PN enters the Poincaré algebra:

$$H = H_N + H_{1PN} + H_{2PN} + H_{SO}^{1PN} + H_{SO}^{2PN} + H_{S^2} + H_{S^3 p} + H_{S^2 p^2} + H_{S^4}$$

- Source terms in canonical variables sufficient for $H_{S_2^2 S_1 p_1}$, $H_{S_2^3 p_1}$, $H_{S_1^3 p_2}$, $H_{S_1^2 S_2 p_2}$, $H_{S_1^2 S_2^2}$, $H_{S_1 S_2^3}$, and $H_{S_2 S_1^3}$ were obtained from the Kerr-metric in ADM coordinates (HS 2007).
- Ansatzes for $H_{S_1^2 p^2}$, $H_{S_2^2 p^2}$, $H_{S_1^3 p_1}$, $H_{S_2^3 p_2}$, $H_{S_1^2 S_2 p_1}$, $H_{S_2^2 S_1 p_2}$, $H_{S_1^4}$, and $H_{S_2^4}$ are fixed up to canonical transformation by $\{G_i, H\} = P_i$.
- The static (linear momentum independent) part of the Hamilton constraint is needed to fix these remaining degrees of freedom.

The Stress-Energy Tensor with Quadrupole

unpublished

- Stress-energy tensor density with quadrupole has the structure:

$$\sqrt{-g}T^{\mu\nu} = \int d\tau \left[t^{\mu\nu}\delta_{(4)} + (t^{\mu\nu\alpha}\delta_{(4)})_{||\alpha} + (t^{\mu\nu\alpha\beta}\delta_{(4)})_{||\alpha\beta} \right]$$

- Getting expressions for the $t^{\mu\nu\cdots}$ from $T^{\mu\nu}{}_{||\nu} = 0$:
  - Dixon's work: Complicated definitions.
  - Tulczyjew's theorems: Complicated calculation.
- $t^{\mu\nu\alpha\beta}$ related to spin-squared mass quadrupole $I_1^{ij} \equiv \gamma^{ik}\gamma^{jl}\gamma^{mn}\hat{S}_{1km}\hat{S}_{1nl} + \frac{2}{3}\mathbf{S}_1^2\gamma^{ij}$.

Ansatz for the Static Source Terms

$$\mathcal{H}^{\text{matter}}_{\mathrm{S}_1^2,\,\text{static}} = \frac{c_1}{m_1}\left(l_1^{ij}\delta_1\right)_{;ij} + \frac{c_2}{m_1}\mathrm{R}_{ij} l_1^{ij}\delta_1 + \frac{c_3}{m_1}\mathbf{S}_1^2\left(\gamma^{ij}\delta_1\right)_{;ij} + \frac{c_4}{m_1}\mathrm{R}\mathbf{S}_1^2\delta_1$$
$$+ \frac{1}{8m_1} g_{mn}\gamma^{pj}\gamma^{ql}\gamma^{mi}{}_{,p}\gamma^{nk}{}_{,q}\hat{S}_{1\,ij}\hat{S}_{1\,kl}\delta_1$$
$$+ \frac{1}{4m_1}\left(\gamma^{ij}\gamma^{mn}\gamma^{kl}{}_{,m}\hat{S}_{1\,ln}\hat{S}_{1\,jk}\delta_1\right)_{,i}$$

- This ansatz is 3-dim. covariant, as p_i is not:

$$p_i = \int d^3\mathbf{x}\,\mathcal{H}^{\text{matter}}_i = m v_i - \frac{1}{2} g_{ij}\gamma^{lm}\gamma^{kj}{}_{,m}\hat{S}_{kl} + \mathcal{O}(p^2) + \mathcal{O}(\hat{S}^2)$$

- Terms like $l_1^{ij}{}_{;k}\delta_1$ or $l_1^{ij}\delta_{1;k}$ can not appear.
- γ_{ij} for Kerr $\Rightarrow c_1 = -\frac{1}{2}$.
- Lapse function for Kerr $\Rightarrow c_2 = 0$.
- c_3 and c_4 do not contribute to the Hamiltonian.

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NLO Spin$_1$-Spin$_1$ Hamiltonian

$$
\begin{aligned}
H^{\text{NLO}}_{S_1^2} = \frac{1}{r_{12}^3}\Bigg[& \frac{m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{S}_1)^2 + \frac{3m_2}{8m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})^2\mathbf{S}_1^2 - \frac{3m_2}{8m_1^3}\mathbf{p}_1^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \\
& - \frac{3m_2}{4m_1^3}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1) - \frac{3}{4m_1 m_2}\mathbf{p}_2^2\mathbf{S}_1^2 \\
& + \frac{9}{4m_1 m_2}\mathbf{p}_2^2(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{S}_1^2 \\
& - \frac{9}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 + \frac{3}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\mathbf{S}_1^2 \\
& - \frac{3}{2m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& + \frac{3}{m_1^2}(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{S}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12}) \\
& - \frac{15}{4m_1^2}(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 \Bigg] \\
- \frac{m_2}{r_{12}^4}\Bigg[& \frac{9}{2}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{5}{2}\mathbf{S}_1^2 + \frac{7m_2}{m_1}(\mathbf{S}_1\cdot\mathbf{n}_{12})^2 - \frac{3m_2}{m_1}\mathbf{S}_1^2 \Bigg]
\end{aligned}
$$

Higher Post-Newtonian Orders

with Han Wang, in preparation

- Need spin corrections to canonical field momentum:

$$\pi^{ij}_{\text{can}} = \pi^{ij}_{\text{field}} + \pi^{ij}_{\text{spin}}\,,$$
$$\pi^{ij}_{\text{field}} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl}\,.$$

- Choose π^{ij}_{spin} such that:

$$P_i = \sum_a p_{ai} - \frac{1}{16\pi}\int d^3x\, \pi^{kl\,\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i}$$
$$J_{ij} = \sum_a (\hat{z}^i_a p_{aj} - \hat{z}^j_a p_{ai}) + \sum_a \hat{S}_{a(i)(j)}$$
$$- \frac{1}{16\pi}\int d^3x\,(x^i \pi^{kl\,\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,j} - x^j \pi^{kl\,\text{TT}}_{\text{can}} h^{\text{TT}}_{kl,i})$$
$$- 2\frac{1}{16\pi}\int d^3x\,(\pi^{ik\,\text{TT}}_{\text{can}} h^{\text{TT}}_{kj} - \pi^{jk\,\text{TT}}_{\text{can}} h^{\text{TT}}_{ki})$$

- Got Hamiltonian for field evolution at formal 3.5PN.
- Checked 1PN energy flux (Kidder 1995).

Action Approach

Steinhoff and Schäfer (2009)

- Action:

$$W[e_{a\mu}, z^{\mu}, \tilde{p}_{\mu}, \Lambda^{Ca}, S_{ab}, \lambda_1^a, \lambda_{2[i]}, \lambda_3] = \int d^4x \, \mathscr{L}$$

$$\Lambda^{Aa}\Lambda^{Bb}\eta_{AB} = \eta^{ab}$$

- Lagrangian densities $\mathscr{L} = \mathscr{L}_G + \mathscr{L}_M + \mathscr{L}_C$:

$$\mathscr{L}_M = \int d\tau \left[\left(\tilde{p}_{\mu} - \frac{1}{2} S_{ab} \, \omega_{\mu}{}^{ab} \right) \frac{dz^{\mu}}{d\tau} + \frac{1}{2} S_{ab} \frac{d\theta^{ab}}{d\tau} \right] \delta_{(4)}$$

$$\mathscr{L}_C = \int d\tau \left[\lambda_1^a \tilde{p}^b S_{ab} + \lambda_{2[i]} \Lambda^{[i]a} \tilde{p}_a - \frac{\lambda_3}{2} (\tilde{p}^2 + m^2) \right] \delta_{(4)}$$

$$\mathscr{L}_G = \frac{1}{16\pi} \sqrt{-g} \mathrm{R}^{(4)}$$

$$d\theta^{ab} = \Lambda_C{}^a d\Lambda^{Cb}$$

Reduction of the Matter Part

in covariant SSC and time gauge

$$\mathscr{L}_M = \mathscr{L}_{MK} + \mathscr{L}_{MC} + \mathscr{L}_{GK}$$

- Kinetic matter part:

$$\mathscr{L}_{MK} = \left[\tilde{p}_i + K_{ij} n S^j + A^{kl} e_{(j)k} e^{(j)}_{l,i} - \left(\frac{1}{2} S_{kj} + \frac{\tilde{p}_{(k} n S_{j)}}{n\tilde{p}}\right) \Gamma^{kj}{}_i\right] \dot{z}^i \delta$$
$$+ \frac{nS^i}{2n\tilde{p}} \dot{\tilde{p}}_i \delta + \left[S_{(i)(j)} + \frac{nS_{(i)}\tilde{p}_{(j)} - nS_{(j)}\tilde{p}_{(i)}}{n\tilde{p}}\right] \frac{\Lambda^{(i)}_{[k]} \dot{\Lambda}^{[k](j)}}{2} \delta$$

- Constraint part $\mathscr{L}_{MC} = N \mathscr{H}^{\text{matter}} - N^i \mathscr{H}^{\text{matter}}_i$:

$$\mathscr{H}^{\text{matter}} = \sqrt{\gamma} T_{\mu\nu} n^\mu n^\nu \,, \qquad \mathscr{H}^{\text{matter}}_i = -\sqrt{\gamma} T_{i\nu} n^\nu$$

Reduction of the Field Part and Result

- Transition to (generalized) Newton-Wigner variables.

$$\Rightarrow \quad \hat{\mathscr{L}}_{GK} = \hat{A}^{ij} e_{(k)i} e_{(k)j,0} \delta \,, \quad g_{ik} g_{jl} \hat{A}^{kl} = \frac{1}{2} \hat{S}_{ij} + \frac{m p_{(i} n S_{j)}}{np(m - np)}$$

- Spatial symmetric gauge (Kibble 1963): $e_{(i)j} = e_{ij} = e_{ji}$

$$e_{ij} e_{jk} = g_{ik} \quad \Rightarrow \quad e_{ij} = \sqrt{g_{ij}}$$

- Definition of field momentum:

$$\pi^{ij}_{\text{can}} = \pi^{ij} + 8\pi \hat{A}^{(ij)} \delta + 16\pi B^{ij}_{kl} \hat{A}^{[kl]} \delta$$

$$e_{k[i} e_{j]k,0} = B^{kl}_{ij} g_{kl,0}$$

- Result:

$$W = \frac{1}{16\pi} \int d^4x \, \pi^{ij\text{TT}}_{\text{can}} h^{\text{TT}}_{ij,0} + \int dt \left[p_i \dot{\hat{z}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \frac{d\hat{\theta}^{(i)(j)}}{dt} - E \right]$$

Thank you for your attention!